

GEOMETRIC PROPERTIES OF THE SHIFTED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We will provide sufficient conditions for the shifted hypergeometric function ${}_2F_1(a, b; c; z)$ to be a member of a specific subclass of starlike functions in terms of the complex parameters a, b and c . For example, we study starlikeness of order α , λ -spirallikeness of order α and strong starlikeness of order α . In particular, those properties lead to univalence of the shifted hypergeometric functions on the unit disk.

1. INTRODUCTION

The (Gaussian) hypergeometric functions appear in various areas in Mathematics, Physics and Engineering and have proved to be quite useful in many respects. Many of the common mathematical functions can be expressed in terms of hypergeometric functions, or suitable limits of them. Their geometric properties in complex domains were, however, studied only recently (in comparison with its long history). For instance, starlikeness and convexity are investigated in 1960's and afterwards. See [14, §4.5], Küstner [9], [10], Hästö, Ponnusamy and Vuorinen [8] and the references therein. This sort of research is not only interesting in the viewpoint of classical analysis, but also applicable in the theory of function spaces, integral transforms, convolutions and so on (see [4], [3] and [8] for example). It should also be noted that most of known results are restricted to real parameter cases. In [21], the authors gave several sufficient conditions for spirallikeness and strong starlikeness of the shifted hypergeometric functions with complex parameters. In the present note, we will extend the results for general classes of starlike functions.

Let \mathcal{A} denote the set of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane and consider the subclasses $\mathcal{A}_0 = \{\varphi \in \mathcal{A} : \varphi(0) = 1\}$ and $\mathcal{A}_1 = \{f \in \mathcal{A} : f(0) = f'(0) - 1 = 0\}$ of \mathcal{A} . A function $f \in \mathcal{A}_1$ is called *starlike* if f is univalent and the image $f(\mathbb{D})$ is starlike with respect to the origin. This property is characterized by the condition $\operatorname{Re} [zf'(z)/f(z)] > 0$ on \mathbb{D} . Robertson [17] refined this notion as follows. For a constant $\alpha \in [0, 1)$, a function $f \in \mathcal{A}_1$ is called *starlike of order α* if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}.$$

As another refinement, Stankiewicz [19] and Brannan and Kirwan [5] introduced independently the class of strongly starlike functions of order α for $0 < \alpha < 1$, which is defined by the condition $|\arg [zf'(z)/f(z)]| < \pi\alpha/2$ on \mathbb{D} . For several geometric characterizations of

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this class, see [20]. It is also known that a strongly starlike function has a quasiconformal extension to the complex plane (see [6]).

As an extension of starlikeness, spirallikeness is natural and useful. For a real number λ with $|\lambda| < \pi/2$, a function $f \in \mathcal{A}_1$ is called λ -spirallike if f is univalent and if the λ -spiral of the form $w \exp(-te^{i\lambda})$, $0 \leq t < +\infty$, is contained in $f(\mathbb{D})$ for each $w \in f(\mathbb{D})$. This is characterized by the condition $\operatorname{Re}[e^{-i\lambda}zf'(z)/f(z)] > 0$ on \mathbb{D} . Libera [12] refined this notion as follows: a function $f \in \mathcal{A}_1$ is called λ -spirallike of order α if

$$\operatorname{Re} \left(e^{-i\lambda} \frac{zf'(z)}{f(z)} \right) > \alpha \cos \lambda, \quad z \in \mathbb{D}.$$

Ma and Minda [13] proposed a unifying way to treat these classes as follows. Let

$$\mathcal{S}^*(\varphi) = \{f \in \mathcal{A}_1 : zf'/f \prec \varphi\}.$$

Here, $\varphi \in \mathcal{A}_0$ and the symbol $g \prec h$ means subordination, that is, $g = h \circ \omega$ for an analytic function ω on \mathbb{D} with $|\omega(z)| \leq |z|$ for $z \in \mathbb{D}$. If h is univalent in \mathbb{D} , $g \prec h$ if and only if $g(0) = h(0)$ and $g(\mathbb{D}) \subset h(\mathbb{D})$. We summarize typical choices of φ and the corresponding classes $\mathcal{S}^*(\varphi)$ in Table 1. See, for instance, [7] for details about the special classes of univalent functions.

$\varphi(z)$	$\mathcal{S}^*(\varphi)$
$\varphi_0(z) = \frac{1+z}{1-z}$	starlike functions
$\varphi_\alpha(z) = \frac{1+(1-2\alpha)z}{1-z}$	starlike functions of order α
$\phi_\alpha(z) = \left(\frac{1+z}{1-z} \right)^\alpha$	strongly starlike functions of order α
$\psi_\lambda(z) = \frac{1+e^{2i\lambda}z}{1-z}$	λ -spirallike functions
$\psi_{\lambda,\alpha}(z) = \frac{1+[e^{2i\lambda}-\alpha(1+e^{2i\lambda})]z}{1-z}$	λ -spirallike functions of order α

TABLE 1. φ and the corresponding class $\mathcal{S}^*(\varphi)$

The main aim in the present paper is to give a sufficient condition for the shifted hypergeometric function $f(z) = {}_2F_1(a, b; c; z)$ to be a member of the class $\mathcal{S}^*(\varphi)$ for a certain φ . To state our main theorem, we introduce a class of admissible functions $Q = \varphi - 1$.

Definition 1. We denote by \mathcal{Q} the set of all analytic functions Q satisfying the following. There exist finitely many points ζ_1, \dots, ζ_n in $\partial\mathbb{D}$ with the following five conditions:

- (i) Q is analytic and univalent on a neighbourhood of $\overline{\mathbb{D}} \setminus \{\zeta_1, \dots, \zeta_n\}$.
- (ii) $Q(0) = 0$.
- (iii) the function $Q(z)$ has a limit w_j in $\widehat{\mathbb{C}}$ as $z \rightarrow \zeta_j$ in \mathbb{D} for each $j = 1, \dots, n$.
- (iv) When $w_j \in \mathbb{C}$, $(Q(z) - w_j)^{\beta_j}$ extends to a univalent function on a neighbourhood of $z = \zeta_j$ for some number $\beta_j > 1$.
- (v) When $w_j = \infty$, $Q(z)^{-\beta_j}$ extends to a univalent function on a neighbourhood of $z = \zeta_j$ for some number $\beta_j \geq 1$.
- (vi) When $w_j = \infty$ and $\beta_j = 1$, the derivative of $P = 1/Q$ satisfies $\zeta_j P'(\zeta_j) \in \mathbb{C} \setminus [0, 1]$.

We note that a similar class was introduced by Miller and Mocanu (see Definition 2.2b in [14]) but our class is more restrictive. It is easy to check the above conditions for all the functions in Table 1 with $0 \leq \alpha < 1$ and $-\pi/2 < \lambda < \pi/2$. As an example, we examine the function $Q = \psi_{\lambda, \alpha} - 1$. In this case $n = 1$ and $\zeta_1 = 1, w_1 = \infty, \beta_1 = 1$. It is enough to check condition (vi) because the other ones are, more or less, obvious. Let $P(z) = 1/Q(z) = (1-z)/[(1-\alpha)(1+e^{2i\lambda})z]$. Then $\zeta_1 P'(\zeta_1) = P'(1) = -1/[(1-\alpha)(1+e^{2i\lambda})]$. If $P'(1) \in [0, 1]$, then $\lambda = 0$ and $-1/(1-\alpha) \in [0, 2]$, which is impossible for $0 \leq \alpha < 1$. Thus, we have seen that $Q \in \mathcal{Q}$ in this case.

The following is our main theorem, from which we will derive several consequences in Sections 3 and 4.

Theorem 1. Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$ and let $\varphi = 1 + Q$ for some $Q \in \mathcal{Q}$. Then $f(z) = z {}_2F_1(a, b; c; z)$ belongs to $\mathcal{S}^*(\varphi)$ if

$$(1.1) \quad 0 < -2 \operatorname{Re} [(pQ(\zeta) + ab)\overline{\zeta Q'(\zeta)}] \quad \text{and}$$

$$(1.2) \quad |B(\zeta)|^2 - |A(\zeta)|^2 \leq -2 \operatorname{Re} [(pQ(\zeta) + ab)\overline{\zeta Q'(\zeta)}]$$

for $\zeta \in \partial\mathbb{D} \setminus \{\zeta_1, \dots, \zeta_n\}$, where $p = a + b + 1 - c$, $A(z) = Q(z)(Q(z) + c - 1)$ and $B(z) = (Q(z) + a)(Q(z) + b)$.

In practical computations, it is convenient to express $|B|^2 - |A|^2$ in the following form:

$$(1.3) \quad |B(z)|^2 - |A(z)|^2 = |w|^2(2 \operatorname{Re} [p\bar{w}] + |a|^2 + |b|^2 - |c - 1|^2) \\ + (2 \operatorname{Re} [a\bar{w}] + |a|^2)(2 \operatorname{Re} [b\bar{w}] + |b|^2),$$

where $w = Q(z)$.

Remark 1. The condition (1.1) can be weakened to $0 \leq -2 \operatorname{Re} [(pQ(\zeta) + ab)\overline{\zeta Q'(\zeta)}]$ if, instead, the condition $A(\zeta) \neq B(\zeta)$ is guaranteed.

Remark 2. We can also obtain a convexity counterpart as follows. Let $\mathcal{K}(\varphi)$ be the class of functions $f \in \mathcal{A}_1$ satisfying $1 + zf''/f' \prec \varphi$ for a given $\varphi \in \mathcal{A}_0$. For $f \in \mathcal{A}_1$, as is well known, $f \in \mathcal{K}(\varphi)$ if and only if $zf' \in \mathcal{S}^*(\varphi)$. When $f(z) = \frac{c}{ab}({}_2F_1(a, b, c; z) - 1)$, by (2.2) below, we have $zf'(z) = z {}_2F_1(a + 1, b + 1, c + 1; z)$. Therefore, we have a sufficient condition for the function $f(z) = \frac{c}{ab}({}_2F_1(a, b, c; z) - 1)$ to be a member of $\mathcal{K}(\varphi)$ as an immediate consequence of Theorem 1, though we do not state it separately.

2. PRELIMINARIES AND PROOF OF THE MAIN THEOREM

First we recall a definition and basic properties of hypergeometric functions. The (Gaussian) *hypergeometric function* ${}_2F_1(a, b; c; z)$ with complex parameters a, b, c ($c \neq 0, -1, -2, \dots$) is defined by the power series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

on $|z| < 1$, where $(a)_n$ is the *Pochhammer symbol*; namely, $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n = 1, 2, \dots$. We note that ${}_2F_1(a, b; c; z)$ analytically continues on the slit plane $\mathbb{C} \setminus [1, +\infty)$. Note here that the hypergeometric function is symmetric in the parameters a and b in the sense that ${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$. As is well known, the hypergeometric function $F(z) = {}_2F_1(a, b; c; z)$ is characterized as the solution to the hypergeometric differential equation

$$(2.1) \quad (1-z)zF''(z) + [c - (a+b+1)z]F'(z) - abF(z) = 0$$

with the initial condition $F(0) = 1$. We also note the derivative formula:

$$(2.2) \quad \frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z).$$

For more properties of hypergeometric functions, we refer to [1], [22] and [23] for example.

As in [11], [21] or in the proof of [18, Theorem 2.12]), our proof of the main theorem will be based on the following (easier) variant of Julia-Wolff theorem (see [16, Prop. 4.13] for instance) and the hypergeometric differential equation (2.1).

Lemma 1. *Let $z_0 \in \mathbb{C}$ with $|z_0| = r \neq 0$ and let ω be a non-constant analytic function on a neighbourhood of $\{z : |z| < r\} \cup \{z_0\}$ with $\omega(0) = 0$. If $|\omega(z)| \leq |\omega(z_0)|$ for $|z| < r$, then $z_0\omega'(z_0) = k\omega(z_0)$ for some $k \geq 1$.*

We are now ready to prove our main theorem.

Proof of Theorem 1. Let ζ_j, w_j and β_j , $j = 1, \dots, n$, be as in Definition 1 and set $F(z) = {}_2F_1(a, b; c; z)$ and $f(z) = zF(z)$. Let us try to show that $zf'(z)/f(z) = 1 + zF'(z)/F(z) \prec \varphi(z) = 1 + Q(z)$. Put $q(z) = zF'(z)/F(z)$. Let $0 < r \leq 1$ be the largest possible number such that $q(z) \in \Omega := Q(\mathbb{D})$ for $|z| < r$. We set $\omega(z) = Q^{-1}(q(z))$ for $|z| < r$. Then $\omega(0) = 0$, $|\omega(z)| < 1$ and $q(z) = Q(\omega(z))$ on $|z| < r$. It thus suffices to show $r = 1$. Suppose, to the contrary, that $r < 1$. Then, there is a $z_0 \in \mathbb{C}$ with $|z_0| = r$ such that $w_0 := q(z_0) \in \partial\Omega$, where the boundary is taken in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We first assume that $w_0 \neq w_j = Q(\zeta_j)$ for $j = 1, \dots, n$. Since $Q(z)$ is univalent near the point $z = \zeta_0 := Q^{-1}(w_0)$, the function $\omega(z)$ extends to $z = z_0$ analytically and satisfies $\omega(z_0) = \zeta_0 \in \partial\mathbb{D}$. Differentiating both sides of the relation $zF'(z) = F(z)Q(\omega(z))$, we get

$$zF''(z) = Q'(\omega(z))\omega'(z)F(z) + [Q(\omega(z)) - 1]F'(z).$$

Substituting it into (2.1) and multiplying with z , we obtain

$$[c - (a+b+1)z + (1-z)(Q(\omega(z)) - 1)]zF'(z) = [(z-1)Q'(\omega(z))\omega'(z) + ab]zF(z).$$

We replace $zF'(z)$ by $F(z)Q(\omega(z))$ in the last formula and rearrange it to obtain

$$[c - (a+b+1)z + (1-z)(Q(\omega(z)) - 1)]F(z)Q(\omega(z)) = [(z-1)Q'(\omega(z))\omega'(z) + ab]zF(z).$$

Since F is not identically zero, we have

$$[c - (a + b + 1)z + (1 - z)(Q(\omega(z)) - 1)]Q(\omega(z)) = [(z - 1)Q'(\omega(z))\omega'(z) + ab]z,$$

which further leads to

$$(2.3) \quad Q(\omega)(Q(\omega) + c - 1) + z\omega'Q'(\omega) = [(Q(\omega) + a)(Q(\omega) + b) + z\omega'Q'(\omega)]z.$$

Lemma 1 now implies that $z_0\omega'(z_0) = k\omega(z_0) = k\zeta_0$ for some $k \geq 1$. Letting $z = z_0$ in (2.3) and using this result, we obtain

$$w_0(w_0 + c - 1) + k\zeta_0Q'(\zeta_0) = [(w_0 + a)(w_0 + b) + k\zeta_0Q'(\zeta_0)]z_0.$$

Let $A = w_0(w_0 + c - 1)$ and $B = (w_0 + a)(w_0 + b)$. In order to get a contradiction, it is enough to show the two conditions:

- (I) $|A + k\zeta_0Q'(\zeta_0)| \geq |B + k\zeta_0Q'(\zeta_0)|$,
- (II) the two equalities $A + k\zeta_0Q'(\zeta_0) = 0$ and $B + k\zeta_0Q'(\zeta_0) = 0$ do not hold simultaneously.

Since $B - A = (a + b + 1 - c)w_0 + ab = pw_0 + ab$, condition (II) follows from the assumption (1.1) (or instead the condition $A(\zeta) \neq B(\zeta)$ as is stated in Remark 1). Condition (I) means that $k\zeta_0Q'(\zeta_0)$ belongs the half-plane $|w + A| \geq |w + B|$. Note that the inequality $|w + A|^2 \geq |w + B|^2$ is equivalent to $|A|^2 - |B|^2 \geq 2 \operatorname{Re}[(B - A)\bar{w}]$. The assumptions (1.1) and (1.2) imply now that $|A|^2 - |B|^2 \geq 2 \operatorname{Re}[(B - A)\overline{\zeta_0Q'(\zeta_0)}] \geq 2 \operatorname{Re}[(B - A)\overline{k\zeta_0Q'(\zeta_0)}]$. Hence, condition (I) follows. In this way, we have excluded the possibility that $w_0 \in \partial\Omega \setminus \{w_1, \dots, w_n\}$.

The remaining possibility is that $w_0 = w_j$ for some j . We first consider the case when $w_0 = w_j \in \mathbb{C}$. By a local property of analytic functions (see [2, Chap.4, §3.3]), $q(z) = h(z)^m + w_j$ near $z = z_0$, where m is a positive integer and $h(z)$ is a univalent analytic function near $z = z_0$ with $h(z_0) = 0$. In particular, the image of the disk $|z| < r$ under q covers a (truncated) sector of opening angle $\pi - \varepsilon$ with vertex at w_j for an arbitrarily small $\varepsilon > 0$. On the other hand, condition (iv) implies that the interior angle of the domain $\Omega = Q(\mathbb{D})$ at w_j is $\pi/\beta_j < \pi$. Therefore, this case does not occur. Next we consider the case when $\zeta_0 = \zeta_j$ and $w_j = \infty$. If $\beta_j > 1$, then the same argument as in the previous case works to conclude that this is impossible. Thus, $\beta_j = 1$. In this case, $P(z) = 1/Q(z)$ is conformal at $z = \zeta_0$, and therefore $P(\zeta_0) = 0$ and $P'(\zeta_0) \neq 0$. Since $Q = 1/P$ and $Q' = -P'/P^2$, the formula (2.3) turns to

$$1 + (c - 1)P(\omega) - z\omega'P'(\omega) = [(1 + aP(\omega))(1 + bP(\omega)) - z\omega'P'(\omega)]z.$$

We now let $z \rightarrow z_0$ to obtain further

$$1 - z_0\omega'(z_0)P'(\zeta_0) = [1 - z_0\omega'(z_0)P'(\zeta_0)]z_0.$$

In view of $|z_0| < 1$ and $z_0\omega'(z_0) = k\omega(z_0) = k\zeta_0$, we conclude that

$$1 - z_0\omega'(z_0)P'(\zeta_0) = 1 - k\zeta_0P'(\zeta_0) = 0,$$

which violates condition (vi). Now all the possibilities have been excluded. We thus conclude that $r = 1$ as required. \square

3. STARLIKENESS AND SPIRALLIKENESS

Note that $\varphi_\alpha = \psi_{0,\alpha}$ and $\psi_\lambda = \psi_{\lambda,0}$ in Table 1. Thus the family $\psi_{\lambda,\alpha}$ covers the cases of starlike functions of order α and λ -spirallike functions. In order to apply Theorem 1 to the function $\psi_{\lambda,\alpha}$, we consider the function

$$Q(z) = \psi_{\lambda,\alpha}(z) - 1 = \frac{(1-\alpha)(1+e^{2i\lambda})z}{1-z} = \frac{2\mu z}{1-z}$$

for $\alpha \in [0, 1)$ and $|\lambda| < \pi/2$, where

$$\mu = (1-\alpha)e^{i\lambda} \cos \lambda.$$

Let $\zeta = e^{i\theta} \in \partial\mathbb{D} \setminus \{1\}$ and $s = \cot \frac{\theta}{2}$ for $0 < \theta < 2\pi$. Simple computations show that

$$\begin{aligned} Q(\zeta) &= \mu \left(\frac{1+\zeta}{1-\zeta} - 1 \right) = \mu(-1+is) \quad \text{and} \\ \zeta Q'(\zeta) &= \frac{2\mu\zeta}{(1-\zeta)^2} = \frac{2\mu}{(e^{i\theta/2} - e^{-i\theta/2})^2} = -\frac{\mu(1+s^2)}{2}. \end{aligned}$$

Now the condition (1.1) is equivalent to the inequality

$$-2 \operatorname{Re} [(pQ(\zeta) + ab)\overline{\zeta Q'(\zeta)}] = (1+s^2) \operatorname{Re} [|\mu|^2(-1+is)p + ab\bar{\mu}] > 0.$$

Since s can be any real number, this inequality forces that $p \in \mathbb{R}$ and (1.1) is further equivalent to $(1+s^2)(\operatorname{Re}[ab\bar{\mu}] - p|\mu|^2) > 0$. By (1.3), we see that $|B(\zeta)|^2 - |A(\zeta)|^2$ is a polynomial in s and

$$|B(\zeta)|^2 - |A(\zeta)|^2 = -2p \operatorname{Im} \mu |\mu|^2 s^3 + O(s^2) \quad (s \rightarrow \pm\infty).$$

Since the right-hand side of (1.2) is a polynomial in s of degree 2, we need the condition $p \operatorname{Im} \mu = 0$ for the inequality (1.2) to hold for all $s \in \mathbb{R}$. Hence, Theorem 1 works only when $\lambda = 0$ or $p = 0$. In this case, the conditions (1.1) and (1.2) reduce to, respectively,

$$(3.1) \quad \operatorname{Re}[ab\bar{\mu}] - p|\mu|^2 > 0 \quad \text{and}$$

$$(3.2) \quad \begin{aligned} &(1+s^2)(\operatorname{Re}[ab\bar{\mu}] - p|\mu|^2) + (2p \operatorname{Re} \mu - |a|^2 - |b|^2 + |c-1|^2)|\mu|^2(1+s^2) \\ &- (2s \operatorname{Im}[a\bar{\mu}] - 2 \operatorname{Re}[a\bar{\mu}] + |a|^2)(2s \operatorname{Im}[b\bar{\mu}] - 2 \operatorname{Re}[b\bar{\mu}] + |b|^2) \geq 0. \end{aligned}$$

Letting $\lambda = 0$, we can prove the following theorem, which gives a sufficient condition for the shifted hypergeometric function to be starlike of order α . Note that this is a natural generalization of [21, Theorem 1.2].

Theorem 2. *Let α be a real constant with $0 \leq \alpha < 1$ and a, b, c be complex numbers with $ab \neq 0$ and $c \neq 0, -1, -2, \dots$. Then the shifted hypergeometric function ${}_2F_1(a, b; c; z)$ is starlike of order α if the following conditions are satisfied:*

- (i) $p = a + b + 1 - c$ is a real number,
- (ii) $\operatorname{Re}[ab] > p(1-\alpha)$,

(iii) $L \geq 0, N \geq 0$ and $LN - M^2 \geq 0$, where

$$\begin{aligned} L &= \frac{\operatorname{Re}[ab]}{1-\alpha} + p(1-2\alpha) - |a|^2 - |b|^2 + |c-1|^2 - 4\operatorname{Im}[a]\operatorname{Im}[b], \\ M &= \frac{\operatorname{Im}[ab(\bar{a} + \bar{b} - 2 + 2\alpha)]}{1-\alpha}, \quad \text{and} \\ N &= \frac{\operatorname{Re}[ab]}{1-\alpha} + p(1-2\alpha) - |a|^2 - |b|^2 + |c-1|^2 \\ &\quad - \left(2\operatorname{Re}[a] - \frac{|a|^2}{1-\alpha}\right) \left(2\operatorname{Re}[b] - \frac{|b|^2}{1-\alpha}\right). \end{aligned}$$

There are several ways to express the coefficients L, M and N . To rephrase them, it is sometimes convenient to use the following elementary formulae:

$$(3.3) \quad \operatorname{Re}[zw] = \operatorname{Re}[z\bar{w}] - 2\operatorname{Im} z \operatorname{Im} w = -\operatorname{Re}[z\bar{w}] + 2\operatorname{Re} z \operatorname{Re} w \quad \text{for } z, w \in \mathbb{C}.$$

Proof. For the choice $\lambda = 0$, we have $\mu = 1 - \alpha$ and $w = (1 - \alpha)(-1 + is)$ in the above observations. For convenience, we write $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Substituting these, the left-hand side of (3.2) can be expressed by

$$\begin{aligned} &(1 - \alpha)^2 \{ \operatorname{Re}[ab]/(1 - \alpha) - p + 2p(1 - \alpha) - |a|^2 - |b|^2 + |c - 1|^2 \} (1 + s^2) \\ &- (1 - \alpha)^2 \{ 2a_2s - 2a_1 + |a|^2/(1 - \alpha) \} \{ 2b_2s - 2b_1 + |b|^2/(1 - \alpha) \} \\ &= (1 - \alpha)^2 (Ls^2 - 2Ms + N). \end{aligned}$$

The above quadratic polynomial in s is non-negative if and only if $L \geq 0, N \geq 0$ and $LN - M^2 \geq 0$. Thus the assertion follows. \square

Let $a = 2(1 - \alpha)$ in Theorem 2. Then $p = b - c + 3 - 2\alpha$ should be real; in other words, $\operatorname{Im} b = \operatorname{Im} c$. Thus, we can consider $f(z) = {}_2F_1(2 - 2\alpha, b + is; c + is; z)$ for real b, c, s . If, in addition, $c - b \geq 0$ and $b + c > 3$, $\operatorname{Re}[a(b + is)] - p(1 - \alpha) = (b + c - 3 + 2\alpha)(1 - \alpha) > 0$ and L, M, N have the simple forms $L = N = 2b + p(1 - 2\alpha) - 4(1 - \alpha)^2 - b^2 + (c - 1)^2 = (c - b)(b + c - 3 + 2\alpha)$, $M = 0$ so that $LN - M^2 = L^2 \geq 0$ and $L = N \geq 0$. Therefore, as a consequence of Theorem 2, we have the next result due to Ruscheweyh [18, Theorem 2.12, p. 60].

Corollary 1. *Let a, b, c, s be real numbers with $0 < a \leq 2$, $3 \leq b + c$ and $b \leq c$. Then the function $f(z) = {}_2F_1(a, b + is; c + is; z)$ is starlike of order $1 - a/2$. In particular, f is starlike.*

Proof. As is accounted above, the assertion follows from Theorem 2 when $3 < b + c$. When $b + c = 3$, we first apply the theorem to the function ${}_2F_1(a, b + \varepsilon + is; c + \varepsilon + is; z)$ for $\varepsilon > 0$ and let $\varepsilon \rightarrow 0$. The assertion follows from the fact that the class $\mathcal{S}^*(\varphi_\alpha)$ is compact (see [15]). \square

Remark 3. The starlikeness of f follows from [21, Theorem 1.2] when $a = 2$ (see [21, Corollary 4.1]). However, it seems that the above corollary does not follow from it for general $a \in (0, 2)$.

We will next apply our main theorem to the case when $p = 0$ and $-\pi/2 < \lambda < \pi/2$. Then we should eliminate c by using the relation $c = a + b + 1$. Our goal is to show the following.

Theorem 3. *Let λ and α be real numbers with $|\lambda| < \pi/2$ and $\alpha \in [0, 1)$. Let a, b be complex numbers with $a + b \neq -1, -2, -3, \dots$. Then the shifted hypergeometric function $f(z) = {}_2F_1(a, b; a + b + 1; z)$ is λ -spirallike of order α if the following two conditions are satisfied:*

- (i) $\operatorname{Re} [e^{-i\lambda} ab] > 0$,
- (ii) $L \geq 0, N \geq 0$ and $LN - M^2 \geq 0$, where

$$\begin{aligned} L &= \operatorname{Re} [e^{-i\lambda} ab(2 - \alpha + (1 - \alpha)e^{-2i\lambda})], \\ M &= \operatorname{Im} [e^{-i\lambda} ab(\bar{a} + \bar{b} - (1 - \alpha)(1 + e^{-2i\lambda}))], \quad \text{and} \\ N &= \operatorname{Re} [e^{-i\lambda} ab(2\bar{a} + 2\bar{b} + \alpha - (1 - \alpha)e^{-2i\lambda})] - |a|^2|b|^2/((1 - \alpha) \cos \lambda). \end{aligned}$$

Remark 4. By [21, Theorem 1.1 (iii)], the condition $\operatorname{Re} [e^{-i\lambda} ab] \geq 0$ is necessary for f to be λ -spirallike.

Proof. For convenience, we put

$$\nu = (1 - \alpha) \cos \lambda.$$

Recalling $p = a + b + 1 - c = 0$ and $w = \mu(-1 + is)$, we see that (3.1) is equivalent to

$$\operatorname{Re} [ab(1 - \alpha)e^{-i\lambda} \cos \lambda] = \nu \operatorname{Re} [abe^{-i\lambda}] > 0.$$

Similarly, the left-hand side of (3.2) is expressed by

$$\begin{aligned} & (1 + s^2) \{ \operatorname{Re} [ab\bar{\mu}] + (|a + b|^2 - |a|^2 - |b|^2)|\mu|^2 \} \\ & - (2s \operatorname{Im} [a\bar{\mu}] - 2 \operatorname{Re} [a\bar{\mu}] + |a|^2) (2s \operatorname{Im} [b\bar{\mu}] - 2 \operatorname{Re} [b\bar{\mu}] + |b|^2) \\ & = ls^2 - 2ms + n, \end{aligned}$$

where

$$\begin{aligned} l &= \operatorname{Re} [ab\bar{\mu}] + 2 \operatorname{Re} [a\bar{\mu}\bar{b}\bar{\mu}] - 4 \operatorname{Im} [a\bar{\mu}] \operatorname{Im} [b\bar{\mu}] \\ &= \operatorname{Re} [ab\bar{\mu}] + 2 \operatorname{Re} [ab\bar{\mu}^2] = \operatorname{Re} [ab\bar{\mu}(1 + 2\bar{\mu})] = \nu L, \\ m &= \operatorname{Im} [a\bar{\mu}](|b|^2 - 2 \operatorname{Re} [b\bar{\mu}]) + \operatorname{Im} [b\bar{\mu}](|a|^2 - 2 \operatorname{Re} [a\bar{\mu}]) \\ &= \operatorname{Im} [ab\bar{\mu}(\bar{a} + \bar{b} - 2\bar{\mu})] = \nu M, \quad \text{and} \\ n &= \operatorname{Re} [ab\bar{\mu}] + 2 \operatorname{Re} [a\bar{\mu}\bar{b}\bar{\mu}] - (|a|^2 - 2 \operatorname{Re} [a\bar{\mu}])(|b|^2 - 2 \operatorname{Re} [b\bar{\mu}]) \\ &= \operatorname{Re} [ab\bar{\mu} - 2ab\bar{\mu}^2 + 2a\bar{b}\bar{\mu} + 2a\bar{a}\bar{b}\bar{\mu}] - |a|^2|b|^2 = \nu N. \end{aligned}$$

Now the assertion follows from Theorem 1. □

When $e^{-i\lambda} ab$ or ab is real, the conditions in the theorem are simplified as follows.

Corollary 2. *Let α and λ be real numbers with $0 \leq \alpha < 1$ and $0 < |\lambda| < \pi/2$. Let a, b be complex numbers with $a + b \neq -1, -2, -3, \dots$ and suppose that $m = e^{-i\lambda} ab$ is a positive*

real number. Then the shifted hypergeometric function ${}_2F_1(a, b; a+b+1; z)$ is λ -spirallike of order α if

$$\begin{aligned} & (\operatorname{Im}[a+b] - (1-\alpha)\sin 2\lambda)^2 \\ & \leq (2-\alpha + (1-\alpha)\cos 2\lambda) \{2\operatorname{Re}[a+b] + \alpha - (1-\alpha)\cos 2\lambda - m/((1-\alpha)\cos \lambda)\}. \end{aligned}$$

Proof. In order to apply Theorem 3, under the assumptions of the corollary, we compute

$$\begin{aligned} L &= m[2-\alpha + (1-\alpha)\cos 2\lambda], \\ M &= -m(\operatorname{Im}[a+b] - (1-\alpha)\sin 2\lambda), \quad \text{and} \\ N &= m[2\operatorname{Re}[a+b] + \alpha - (1-\alpha)\cos 2\lambda - m/((1-\alpha)\cos \lambda)]. \end{aligned}$$

Thus the assertion follows. \square

We note that $L > 0$ and $M^2 \leq LN$ both implies that $N \geq 0$ in the assumption of Theorem 3. We obtain the next result by keeping it in mind.

Corollary 3. *Let α and λ be real numbers with $0 \leq \alpha < 1$ and*

$$\frac{1-2\alpha}{4(1-\alpha)} < \cos^2 \lambda < 1.$$

Let a, b be complex numbers with $a+b \neq -1, -2, -3, \dots$ and suppose that $m = ab$ is a positive real number. Then the shifted hypergeometric function ${}_2F_1(a, b; a+b+1; z)$ is λ -spirallike of order α if

$$\begin{aligned} & \left(\frac{\operatorname{Im}[e^{i\lambda}(a+b)]}{\cos \lambda} - 2(1-\alpha)\sin 2\lambda \right)^2 \leq (4(1-\alpha)\cos^2 \lambda + 2\alpha - 1) \\ & \times \left(\frac{2\operatorname{Re}[e^{i\lambda}(a+b)]}{\cos \lambda} - 4(1-\alpha)\cos^2 \lambda + (3-2\alpha) - \frac{m}{(1-\alpha)\cos^2 \lambda} \right) \end{aligned}$$

Proof. Similarly, assuming that $m = ab$ is positive, we compute

$$\begin{aligned} L &= m \operatorname{Re}[e^{-i\lambda}(2-\alpha + (1-\alpha)e^{-2i\lambda})] = m \cos \lambda [4(1-\alpha)\cos^2 \lambda - (1-2\alpha)], \\ M &= m \operatorname{Im}[e^{-i\lambda}(\bar{a} + \bar{b} - 2(1-\alpha)e^{-i\lambda}\cos \lambda)] \\ &= -m[\operatorname{Im}[e^{i\lambda}(a+b)] - 2(1-\alpha)\sin 2\lambda \cos \lambda], \quad \text{and} \\ N &= m \operatorname{Re}[e^{-i\lambda}(2\bar{a} + 2\bar{b} + \alpha - (1-\alpha)e^{-2i\lambda})] - m^2/((1-\alpha)\cos \lambda) \\ &= m\{2\operatorname{Re}[e^{i\lambda}(a+b)] + \alpha \cos \lambda - (1-\alpha)\cos 3\lambda - m/((1-\alpha)\cos \lambda)\}. \end{aligned}$$

Here, we used the formula $\cos 3\lambda = \cos \lambda(4\cos^2 \lambda - 3)$. Observe that $L > 0$ precisely if $\cos^2 \lambda > (1-2\alpha)/4(1-\alpha)$. The assertion now follows from Theorem 3. \square

When $\alpha = 0$, Theorem 3 and Corollaries 2 and 3 reduce to Theorem 1.4 and Corollaries 1.5 and 1.6 in [21], respectively.

4. STRONG STARLIKENESS

The authors gave in the previous paper [21] a sufficient condition for the shifted hypergeometric function to be strongly starlike as in the following.

Theorem A ([21]). *Let $1/3 < \alpha < 1$ and a, b be complex numbers with $a + b \in \mathbb{R}$ and $ab > 0$. Then the shifted hypergeometric function ${}_2F_1(a, b; a + b + 1; z)$ is strongly starlike of order α if*

$$\{(a - b)^2 + 6(a + b) - 3\} \sin^2 \frac{\pi\alpha}{2} \geq a^2 + ab + b^2.$$

However, this was obtained as a corollary of the spirallikeness result (Theorem 3 with $\alpha = 0$). Therefore, the unpleasant assumption $1/3 < \alpha < 1$ was inevitable. The following can be obtained as a consequence of the main theorem. Though the condition is much involved, this restriction does not appear explicitly.

Theorem 4. *Let α be a real number with $0 < \alpha < 1$ and let a, b, c be complex numbers satisfying $ab \neq 0$ and $c \neq 0, -1, -2, \dots$. The shifted hypergeometric function $f(z) = {}_2F_1(a, b; c; z)$ is strongly starlike of order α if the following three conditions are satisfied:*

- (i) $p = a + b + 1 - c$ is a real number,
- (ii) $|\arg(ab - p)| < \pi\alpha/2$,
- (iii) $G_\varepsilon(s^\alpha) \leq \alpha(s + s^{-1})s^\alpha \operatorname{Re} [(ab - p)e^{\varepsilon\pi i(1-\alpha)/2}]$, for $s \in (0, +\infty)$, $\varepsilon = \pm 1$, where $G_\varepsilon(x) = Sx^3 + T_\varepsilon x^2 + U_\varepsilon x + V$ with

$$S = 2p \cos \frac{\pi\alpha}{2},$$

$$T_\varepsilon = |a|^2 + |b|^2 - |c - 1|^2 - 2p - 4p \cos^2 \frac{\pi\alpha}{2} + 4 \operatorname{Re} [ae^{-\varepsilon i\pi\alpha/2}] \operatorname{Re} [be^{-\varepsilon i\pi\alpha/2}],$$

$$U_\varepsilon = -2(|a|^2 + |b|^2 - |c - 1|^2 - 3p) \cos \frac{\pi\alpha}{2} \\ + 2 \operatorname{Re} [ae^{-\varepsilon i\pi\alpha/2}] (|b|^2 - 2 \operatorname{Re} b) + 2 \operatorname{Re} [be^{-\varepsilon i\pi\alpha/2}] (|a|^2 - 2 \operatorname{Re} a),$$

$$V = |a|^2 + |b|^2 - |c - 1|^2 - 2p + (|a|^2 - 2 \operatorname{Re} a)(|b|^2 - 2 \operatorname{Re} b).$$

Proof. To prove that $f \in \mathcal{S}^*(\phi_\alpha)$, it is sufficient to check the conditions in Theorem 1 for

$$Q(z) = \phi_\alpha(z) - 1 = \left(\frac{1+z}{1-z} \right)^\alpha - 1$$

with $\alpha \in [0, 1)$.

Let $\zeta = e^{i\theta}$ for $0 < |\theta| < \pi$, $\varepsilon = \theta/|\theta| = \operatorname{sgn} \theta$, and $s = \cot(|\theta|/2) \in (0, +\infty)$. As in the previous section, we have

$$Q(\zeta) = (\varepsilon i s)^\alpha - 1 = e^{i\beta} s^\alpha - 1 \quad \text{and} \\ \zeta Q'(\zeta) = [Q(\zeta) + 1] \frac{2\alpha e^{i\theta}}{1 - e^{2i\theta}} = -\frac{\alpha e^{i\beta} s^\alpha (1 + s^2)}{2i\varepsilon s} = -\frac{\alpha}{2} e^{-i\gamma} s^\alpha (s + s^{-1}),$$

where

$$\beta = \varepsilon \frac{\pi\alpha}{2} \quad \text{and} \quad \gamma = \varepsilon \frac{\pi(1-\alpha)}{2}.$$

Hence by using these relations, we get

$$\begin{aligned} -2 \operatorname{Re} [(pQ(\zeta) + ab)\overline{\zeta Q'(\zeta)}] &= \operatorname{Re} [(pe^{i\beta}s^\alpha + ab - p)\alpha e^{i\gamma}s^\alpha(s + s^{-1})] \\ &= \alpha(s + s^{-1})s^\alpha \{ -\varepsilon s^\alpha \operatorname{Im} p + \operatorname{Re} [e^{i\gamma}(ab - p)] \}. \end{aligned}$$

By (1.1), we need the conditions

$$\operatorname{Im} p = 0 \quad \text{and} \quad \operatorname{Re} [e^{i\gamma}(ab - p)] > 0$$

for $\varepsilon = \pm 1$. In this way, we arrived at the first and second conditions in the theorem. From now on, we assume that these two conditions are satisfied. In view of (1.3), we compute

$$\begin{aligned} &|B(\zeta)|^2 - |A(\zeta)|^2 \\ &= (s^{2\alpha} - 2s^\alpha \cos \beta + 1)(2ps^\alpha \cos \beta - 2p + |a|^2 + |b|^2 - |c - 1|^2) \\ &\quad + (2 \operatorname{Re} [ae^{-i\beta}s^\alpha - a] + |a|^2)(2 \operatorname{Re} [be^{-i\beta}s^\alpha - b] + |b|^2) \\ &= G_\varepsilon(s^\alpha). \end{aligned}$$

Therefore the condition (1.2) can be presented as in the third condition of the theorem. \square

We now assume that $p = a + b + 1 - c = 0$. Letting $\eta = e^{-\varepsilon i\pi\alpha/2}$, we compute the coefficients of $G_\varepsilon(x)$ in Theorem 4 by making use of (3.3) as follows: $S = 0$,

$$\begin{aligned} T_\varepsilon &= -2 \operatorname{Re} [a\bar{b}] + 4 \operatorname{Re} [a\eta] \operatorname{Re} [b\eta] = 2 \operatorname{Re} [ab\eta^2], \\ U_\varepsilon &= 2 \operatorname{Re} [a\bar{b}](\eta + \bar{\eta}) + 2 \operatorname{Re} [a\eta(b\bar{b} - b - \bar{b}) + b\eta(a\bar{a} - a - \bar{a})] \\ &= 2 \operatorname{Re} [ab\eta(\bar{a} + \bar{b} - 2)], \\ V &= -2 \operatorname{Re} [a\bar{b}] + |ab|^2 - 2 \operatorname{Re} [ab\bar{b} + a\bar{a}b] + 4 \operatorname{Re} a \operatorname{Re} b \\ &= 2 \operatorname{Re} [ab] + |ab|^2 - 2 \operatorname{Re} [ab(\bar{a} + \bar{b})]. \end{aligned}$$

Thus, we obtain the following corollary.

Corollary 4. *Let α be a real number with $0 < \alpha < 1$ and let a, b be complex numbers with $a + b \neq -1, -2, -3, \dots$. The shifted hypergeometric function ${}_2F_1(a, b; a + b + 1; z)$ is strongly starlike of order α if*

- (i) $|\arg(ab)| < \pi\alpha/2$, and
- (ii)

$$\begin{aligned} &2s^{2\alpha} \operatorname{Re} [e^{-\varepsilon\pi i\alpha} ab] + 2s^\alpha \operatorname{Re} [e^{-\varepsilon\pi i\alpha/2} ab(\bar{a} + \bar{b} - 2)] \\ &+ |ab|^2 - 2 \operatorname{Re} [ab(\bar{a} + \bar{b} - 1)] \leq \alpha(s + s^{-1})s^\alpha \operatorname{Re} [e^{\varepsilon\pi i(1-\alpha)/2} ab] \end{aligned}$$

for all $s \in (0, +\infty)$ and both signs $\varepsilon = \pm 1$.

Since the condition in the last corollary is still implicit, we make a crude estimate. We first prepare the following elementary lemma.

Lemma 2. *Let α, A, B, C and K be constants with $0 < \alpha < 1$ and $K > 0$. If $B/2 + \max\{A, C\} \leq K$, then*

$$As^\alpha + B + Cs^{-\alpha} \leq K(s + s^{-1}) \quad \text{for } s \in (0, +\infty).$$

Proof. It is easy to observe that $s^\alpha + s^{-\alpha}$ is increasing in $0 \leq \alpha \leq 1$ for a fixed $s > 0, s \neq 1$. In particular, $2 < s^\alpha + s^{-\alpha} < s + s^{-1}$ for $0 < \alpha < 1, s > 0, s \neq 1$. Thus the assertion follows. \square

We can now deduce the following from Corollary 4.

Corollary 5. *Let α be a real number with $0 < \alpha < 1$. Assume that complex numbers a, b with $a + b \neq -1, -2, -3, \dots$ satisfy the conditions:*

- (i) $|\arg(ab)| < \pi\alpha/2$, and
- (ii) $\operatorname{Re}[e^{-\varepsilon\pi i\alpha/2}ab(\bar{a} + \bar{b} - 2)] + \max\{2\operatorname{Re}[e^{-\varepsilon\pi i\alpha}ab], |ab|^2 - 2\operatorname{Re}[ab(\bar{a} + \bar{b} - 1)]\} \leq \alpha \operatorname{Re}[e^{\varepsilon\pi i(1-\alpha)/2}ab], \text{ for } \varepsilon = \pm 1.$

Then the function $z_2F_1(a, b; a + b + 1; z)$ is strongly starlike of order α .

We further assume that $l = a + b$ is real and $m = ab$ is real and positive. Then the condition (ii) in the last corollary reads

$$(l - 2) \cos \frac{\pi\alpha}{2} + \max\{2 \cos \pi\alpha, m - 2(l - 1)\} \leq \alpha \sin \frac{\pi\alpha}{2}.$$

In particular, if $2 \cos \pi\alpha \geq m - 2(l - 1)$, or equivalently, if $m - 2l + 4 = (a - 2)(b - 2) \leq 2 \cos \pi\alpha + 2 = 4 \cos^2(\pi\alpha/2)$, the condition takes the form $(l - 2) \cos(\pi\alpha/2) + 2 \cos \pi\alpha \leq \alpha \sin(\pi\alpha/2)$. Therefore, we finally obtain the following corollary.

Corollary 6. *Let α be a real number with $0 < \alpha < 1$ and assume that a, b are complex numbers such that $a + b$ is real and $ab > 0$. If, in addition,*

$$(a - 2)(b - 2) \leq 4 \cos^2 \frac{\pi\alpha}{2} \quad \text{and} \quad a + b \leq 2 - 2 \frac{\cos \pi\alpha}{\cos \frac{\pi\alpha}{2}} + \alpha \tan \frac{\pi\alpha}{2},$$

then the function $z_2F_1(a, b; a + b + 1; z)$ is strongly starlike of order α .

We remark that, under the hypothesis in the last corollary, $l = a + b$ should satisfy the inequalities

$$2 \sin^2 \frac{\pi\alpha}{2} < \frac{ab}{2} + 2 \sin^2 \frac{\pi\alpha}{2} \leq l \leq 2 - 2 \frac{\cos \pi\alpha}{\cos \frac{\pi\alpha}{2}} + \alpha \tan \frac{\pi\alpha}{2}.$$

We can see that $[2 - 2 \cos \pi\alpha / \cos(\pi\alpha/2) + \alpha \tan(\pi\alpha/2)] - 2 \sin^2(\pi\alpha/2)$ is positive for $0 < \alpha < 1$. Therefore, there are some $a, b \in \mathbb{C}$ satisfying the hypothesis in the corollary for each $0 < \alpha < 1$. Compare with Theorem A.

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